On the stability of the secular evolution of the planar Sun–Jupiter–Saturn–Uranus system *

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Abstract

We investigate the long time stability of the Sun–Jupiter–Saturn–Uranus system by considering the planar, secular model. Our method may be considered as an extension of Lagrange's theory for the secular motions. Indeed, concerning the planetary orbital revolutions, we improve the classical circular approximation by replacing it with a torus which is invariant up to order two in the masses; therefore, we investigate the stability of the elliptic equilibrium point of the secular system for small values of the eccentricities. For the initial data corresponding to a real set of astronomical observations, we find an estimated stability time of 10^7 years, which is not extremely smaller than the lifetime of the Solar System (~ 5 Gyr).

1 Introduction

In this paper we revisit the problem of the stability of the Solar System, at least considering (some of) the major planets, in the light of both Kolmogorov's and Nekhoroshev's

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theories. Some aspects are also related to the theory of Lagrange and Laplace on the secular motions of perihelia and nodes of the planetary orbits.

One of our main aims is to point out the major dynamical and computational difficulties that arise in the application of Kolmogorov's theorem. In view of this, we attempt to apply the Nekhoroshev's theory by trying essentially an extension of Lagrange's theory. Although the final results appear to be interesting, our conclusion will be that further and more refined investigations are needed. We consider indeed the present paper as the beginning of a more comprehensive study of systems with more than two planets in the framework of perturbation methods related to the theories above.

In 1954 Kolmogorov announced his celebrated theorem on the persistence under small perturbation of quasi periodic motions on invariant tori of an integrable Hamiltonian systems (see [22]). The relevance of that result for the problem of stability of the Solar System was pointed out by Kolmogorov himself, and later emphasized in the subsequent papers of Moser (see [39]) and Arnold (see [1]). The three papers mentioned above marked the beginning of the so called KAM theory.

However the actual applicability of Kolmogorov's theorem to the planetary system encounters two major difficulties, namely: (i) the degeneracy of the Keplerian motion, and (ii) the extremely restrictive assumptions on the smallness of the perturbation.

The former difficulty is related to the elliptic form of the Keplerian orbits. Indeed a system including a central body (a star) and $n > 1$ planets, after elimination of the known first integrals, has $3n-2$ degrees of freedom, while only n actions appear in the Keplerian part of the Hamiltonian. The way out proposed by Arnold, and inspired by the approach of Lagrange and Laplace, was to introduce in the proof two separate time scales for the orbital motion and for the secular evolution of the perihelia and of the nodes (see [2] and its recent extension in [9]). Such an approach has been successfully extended to the $n+1$–body planetary systems thanks to the work done by Herman and Féjoz (see [11]).

Attacking the second difficulty (i.e., the unrealistic requirements on the smallness of masses, eccentricities and inclinations of the planets) with purely analytical methods seems to be unrealistic. However, some positive results could be attained using computer algebra. This means that we explicitly perform a few perturbation steps, thus getting an approximation of the wanted invariant torus which is good enough to allow us to apply analytical methods. Such an approach (also implementing interval arithmetic) allowed some authors to rigorously prove the existence of KAM tori for some interesting problems in Celestial Mechanics (see, e.g., [6], [7], [8], [35] and [12]). However, all these works consider models having just two degrees of freedom. This because increasing the number of independent variables makes the explicit calculation of perturbation steps a big challenge, due to the dramatic increase of the number of coefficients to be calculated, so that a sufficiently good initial approximation of an invariant torus is hardly obtained. For what concerns problems with more than two degrees of freedom, in a few cases only the availability of an algorithmic version of Kolmogorov's theorem (see [3], [17] and [18]) allowed us to obtain a good approximation of the invariant tori, although this approach is not yet sufficient for a fully rigorous application of the theory. For instance, the constructed solution on a KAM torus has been successfully compared with the real motion of the Sun–Jupiter–Saturn system, which can be represented by a model with 4 degrees

of freedom (see [37] for all details). Moreover our recent work focuses on a first study of the long time stability in a neighborhood of such KAM torus (see [19]).

Besides the technical difficulty, the results of the numerical explorations have raised some doubts concerning the applicability of Kolmogorov's theory to the major bodies of our planetary system, namely the Sun and the so called Jovian planets, i.e. Jupiter, Saturn, Uranus and Neptune, hereafter we will refer to this model as the SJSUN problem. Indeed, the motion of such planetary subsystem has been shown to be chaotic by Sussman and Wisdom (see [49]). Murray and Holman provided such an enlightening explanation of this phenomenon, that we think it is helpful to briefly summarize some of their results as follows (see [40] for completeness).

(a) The chaoticity of the Jovian planets appears to be due to the *overlap of some resonances involving three or four bodies*. An example is given by the resonances

$$3n_1 - 5n_2 - 7n_3 + [(3-j)g_1 + 6g_2 + jg_3] \ , \qquad \text{with } j = 0,\, 1,\, 2,\, 3 \ ,$$

where n_i stands for the mean motion frequency of the i–th planet, g_i means the (secular) frequency of its perihelion argument and the indexes 1, 2, 3 refer to Jupiter, Saturn and Uranus, respectively. In fact, during the planetary motion each angle corresponding to the resonances above moves from libration to rotation and viceversa. Many other resonances analogous to the previous ones are located in the vicinity of the real orbit of the SJSUN system, some of them involving also Neptune and the frequencies related to the longitudes of the nodes.

(b) The time needed by these resonances to eject Uranus from the Solar System is roughly evaluated to be about 10^{18} years.

(c) By moving the initial semi–major axis of Uranus in the range 19.18–19.35 AU one observes some regions that look filled by quasi–periodic ordered motions and other regions that are weakly chaotic, i.e., with a Lyapunov time ranging between 2×10^5 and 10^8 years. All the main resonances acting in this region involve the linear combination $3n_1 - 5n_2 - 7n_3$ among the mean motion frequencies of Jupiter, Saturn and Uranus.

(d) The result (c) qualitatively persists also for the *planar* SJSUN system or when the influence of Neptune is neglected.

(e) Conversely, no chaotic motions are detected in the *planar* system including the Sun, Jupiter, Saturn and Uranus (hereafter, SJSU for shortness) for the same initial values of the semi–major axis of Uranus considered at point (c). This suggests that the resonances described at point (a) affect observable regions only when combined with some effects induced by Neptune or by the mutual inclinations.

By the way, we note that the resonances involving the linear combination $3n_1 - 5n_2 - 7n_3$ are clearly related to the approximate ratio 5 : 2 and 7 : 1 between the orbital motion of Jupiter and Saturn and of Jupiter and Uranus, respectively. Similarly, the ratio 2 : 1 between Uranus and Neptune appears also to be relevant (historically, this helped Le Verrier to predict the existence and the location of Neptune). The low order

of the latter resonance may explain why the influence of Neptune induces some chaotic behaviour, as pointed out in (d) and (e) above.

The weak chaos in the motion of the Jovian planets makes somehow hopeless the task of describing their long–term evolution by a quasi–periodic approximation, as it is provided by the KAM theory. Therefore it appears to be more natural to look for exponential stability as assured by Nekhoroshev's theory (see [41] and [42]). Indeed the theorem of Nekhoroshev applies to an open set of initial conditions, and states that the stability time increases exponentially with the inverse of the perturbation parameter. Our aim is to investigate whether the SJSU system may remain close to its current conditions for a time that exceeds the lifetime of the system itself; e.g., in our case the age of the Universe, which is estimated to be $\sim 1.4\times 10^{10}$ years, could be enough. We stress that the rather long time reported in (d) concerning the possible dissolution of the SJSUN system seems to support our hope. The approach based on Nekhoroshev's theory has been applied during the last decades to the case of the Trojan asteroids, producing realistic results (see, e.g., [21], [47], [10] and [33]). Concerning the SJSUN system, we expect that a combination of both the KAM and the Nekhoroshev theory could prove that the motion remains close to an invariant torus for very long times (see [38] and [19]).

In the present paper, we restrict our attention to the SJSU *planar* system, due to the huge computational difficulties one encounters during the expansion of the Hamiltonian. Indeed, a rather long preliminary work is necessary in order to give the Hamiltonian a convenient form for starting more standard perturbation methods (see [35], [36] and [37]). We devote sects. 2 and 3.1 to this part of the problem.

Furthermore, in the line of Lagrange's theory, we focus only on the secular part of the Hamiltonian, which is derived in subsect. 3.2. Let us emphasize that all along both sects. 2 and 3 we pay a special attention to include all the relevant terms related to the three–body mean motion quasi–resonance $3n_1-5n_2-7n_3$ in view of the remarks reported at points (a) and (c) above).

The secular system turns out to have the form of a perturbed system of harmonic oscillators. It can be remarked that the reduction to the plane model makes it possible to investigate the stability of the equilibrium using the theorem of Dirichlet. However, this is enough if we restrict our attention to the planar secular model, but does not apply neither to the spatial case, due to the opposite signs in the secular frequencies of the perihelia and of the nodes, nor in the full non secular model. Thus, we think that it is also useful to proceed by investigating the stability in Nekhoroshev's sense, since this may give indication about the possibility of extending our calculation to more refined models. This part is worked out in sect. 4.

Finally, sect. 5 is devoted to the conclusions.

2 Classical expansion of the planar planetary Hamiltonian

Let us consider four point bodies P_0, P_1, P_2, P_3, with masses m_0, m_1, m_2, m_3, mutually interacting according to Newton's gravitational law. Hereafter the indexes 0, 1, 2, 3 will

correspond to Sun, Jupiter, Saturn and Uranus, respectively.

Let us now recall how the classical Poincaré variables can be introduced so to perform a first expansion of the Hamiltonian around circular orbits, i.e., having zero eccentricity. We basically follow the formalism introduced by Poincaré (see [43] and [44]; for a modern exposition, see, e.g., [30] and [32]). We remove the motion of the center of mass by using heliocentric coordinates $\underline{r}_j = \overrightarrow{P_0 P_j}$, with $j = 1, 2, 3$. Denoting by $\underline{\tilde{r}}_j$ the momenta conjugated to $\underline{r}_j$, the Hamiltonian of the system has 6 degrees of freedom, and reads

$$F(\underline{\tilde{r}}, \underline{r}) = T^{(0)}(\underline{\tilde{r}}) + U^{(0)}(\underline{r}) + T^{(1)}(\underline{\tilde{r}}) + U^{(1)}(\underline{r}) \ , \tag{1}$$

where

$$\begin{aligned} T^{(0)}(\underline{\tilde{r}}) &= \tfrac{1}{2} \sum_{j=1}^{3} \tfrac{m_0 + m_j}{m_0 m_j} \|\underline{\tilde{r}}_j\|^2 \ , & T^{(1)}(\underline{\tilde{r}}) &= \tfrac{1}{m_0} \Big(\underline{\tilde{r}}_1 \cdot \underline{\tilde{r}}_2 + \underline{\tilde{r}}_1 \cdot \underline{\tilde{r}}_3 + \underline{\tilde{r}}_2 \cdot \underline{\tilde{r}}_3 \Big) \ , \\ U^{(0)}(\underline{r}) &= -\mathcal{G} \sum_{j=1}^{3} \tfrac{m_0 \, m_j}{\|\underline{r}_j\|} \ , & U^{(1)}(\underline{r}) &= -\mathcal{G} \left(\tfrac{m_1 \, m_2}{\|\underline{r}_1 - \underline{r}_2\|} + \tfrac{m_1 \, m_3}{\|\underline{r}_1 - \underline{r}_3\|} + \tfrac{m_2 \, m_3}{\|\underline{r}_2 - \underline{r}_3\|} \right) \ . \end{aligned}$$

The plane set of Poincaré's canonical variables is introduced as

$$\begin{aligned} \Lambda_j &= \frac{m_0 \, m_j}{m_0 + m_j} \sqrt{\mathcal{G}(m_0 + m_j) a_j} \ , & \lambda_j &= M_j + \omega_j \ , \\ \xi_j &= \sqrt{2\Lambda_j} \sqrt{1 - \sqrt{1 - e_j^2}} \, \cos\omega_j \ , & \eta_j &= -\sqrt{2\Lambda_j} \sqrt{1 - \sqrt{1 - e_j^2}} \, \sin\omega_j \ , \end{aligned} \tag{2}$$

for $j = 1\,, 2\,, 3\,,$ where a_j , e_j , M_j and ω_j are the semi–major axis, the eccentricity, the mean anomaly and the perihelion argument, respectively, of the j–th planet. One immediately sees that both ξ_j and η_j are of the same order of magnitude as the eccentricity e_j .

Using Poincaré's variables (2), the Hamiltonian F can be rearranged so that one has

$$F(\underline{\Lambda}, \underline{\lambda}, \underline{\xi}, \underline{\eta}) = F^{(0)}(\underline{\Lambda}) + \mu F^{(1)}(\underline{\Lambda}, \underline{\lambda}, \underline{\xi}, \underline{\eta}) \ , \tag{3}$$

where $F^{(0)} = T^{(0)} + U^{(0)}$, $\mu F^{(1)} = T^{(1)} + U^{(1)}$. Here, the small dimensionless parameter $\mu = \max\{m_1 \,/\, m_0 \,,\, m_2 \,/\, m_0 \,,\, m_3 \,/\, m_0\, \}$ has been introduced in order to highlight the different size of the terms appearing in the Hamiltonian. Let us remark that the time derivative of each coordinate is $\mathcal{O}(\mu)$ but in the case of the angles $\underline{\lambda}$. Therefore, according to the common language in Celestial Mechanics, in the following we will refer to $\underline{\lambda}$ and to their conjugate actions $\underline{\Lambda}$ as the *fast variables*, while $(\underline{\xi}, \underline{\eta})$ will be called *secular variables.*

We proceed now by expanding the Hamiltonian (3) in order to construct the first basic approximation of Kolmogorov's normal form. We pick a value $\underline{\Lambda}^*$ for the fast actions and perform a translation $\mathcal{T}_{\underline{\Lambda}^*}$ defined as

$$L_j = \Lambda_j - \Lambda_j^* \ , \qquad \text{for } j = 1\,, 2\,, 3\,. \tag{4}$$

This is a canonical transformation that leaves the coordinates $\underline{\lambda}$, $\underline{\xi}$ and $\underline{\eta}$ unchanged. The transformed Hamiltonian $\mathcal{H}^{(\mathcal{T})} = F \circ \mathcal{T}_{\underline{\Lambda}^*}$ can be expanded in power series of $\underline{L}$, $\underline{\xi}$, $\underline{\eta}$

Table 1: Masses m_j and initial conditions for Jupiter, Saturn and Uranus in our planar model. We adopt the UA as unit of length, the year as time unit and set the gravitational constant $\mathcal{G} = 1$. With these units, the solar mass is equal to $(2\pi)^2$. The initial conditions are expressed by the usual heliocentric planar orbital elements: the semi–major axis a_j , the mean anomaly M_j , the eccentricity e_j and the perihelion longitude ω_j . The data are taken by JPL at the Julian Date 2440400.5 .

	Jupiter ($j = 1$)	Saturn ($j = 2$)	Uranus ($j = 3$)
m_j	$(2\pi)^2/1047.355$	$(2\pi)^2/3498.5$	$(2\pi)^2/22902.98$
a_j	5.20463727204700266	9.54108529142232165	19.2231635458410572
M_j	3.04525729444853654	5.32199311882584869	0.19431922829271914
e_j	0.04785365972484999	0.05460848595674678	0.04858667407651962
ω_j	0.24927354029554571	1.61225062288036902	2.99374344439246487

around the origin. Thus, forgetting an unessential constant we rearrange the Hamiltonian of the system as

$$\mathcal{H}^{(\mathcal{T})}(\underline{L}, \underline{\lambda}, \underline{\xi}, \underline{\eta}) = \underline{n}^* \cdot \underline{L} + \sum_{j_1=2}^{\infty} h_{j_1,0}^{(\mathrm{Kep})}(\underline{L}) + \mu \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} h_{j_1,j_2}^{(\mathcal{T})}(\underline{L}, \underline{\lambda}, \underline{\xi}, \underline{\eta}) \ , \tag{5}$$

where the functions $h_{j_1,j_2}^{(\mathcal{T})}$ are homogeneous polynomials of degree j_1 in the actions $\underline{L}$ and of degree j_2 in the secular variables $(\underline{\xi}, \underline{\eta})$. The coefficients of such homogeneous polynomials do depend analytically and periodically on the angles $\underline{\lambda}$. The terms $h_{j_1,0}^{(\mathrm{Kep})}$ of the Keplerian part are homogeneous polynomials of degree j_1 in the actions $\underline{L}$, the explicit expression of which can be determined in a straightforward manner. In the latter equation the term which is both linear in the actions and independent of all the other canonical variables (i.e., $\underline{n}^* \cdot \underline{L}$) has been separated in view of its relevance in perturbation theory, as it will be discussed in the next section. We also expand the coefficients of the power series $h_{j_1,j_2}^{(\mathcal{T}_F)}$ in Fourier series of the angles $\underline{\lambda}$. The expansion of the Hamiltonian is a traditional procedure in Celestial Mechanics. We work out these expansions for the case of the planar SJSU system using a specially devised algebraic manipulation. The calculation is based on the approach described in sect. 2.1 of [35], which in turn uses the scheme sketched in sect. 3.3 of [46].

The reduction to the planar case is performed as follows. We pick from Table IV of [48] the initial conditions of the planets in terms of heliocentric positions and velocities at the Julian Date 2440400.5 . Next, we calculate the corresponding orbital elements with respect to the invariant plane (that is perpendicular to the total angular momentum). Finally we include the longitudes of the nodes Ω_j (which are meaningless in the planar case) in the corresponding perihelion longitude ω_j and we eliminate the inclinations by setting them equal to zero. The remaining initial values of the orbital elements are reported in Table 1.

Having determined the initial conditions we come to determining the average values $(a_1^*\, , \, a_2^*\, , \, a_3^*)$ of the semi–major axes during the evolution. To this end we perform a

long–term numerical integration of Newton's equations starting from the initial conditions related to the data reported in Table 1. After having computed $(a_1^*\,,\,a_2^*\,,\,a_3^*)$, we determine the values $\underline{\Lambda}^*$ via the first equation in (2). This allows us to perform the expansion (5) of the Hamiltonian as a function of the canonical coordinates $(\underline{L},\underline{\lambda},\underline{\xi},\underline{\eta})$. In our calculations we truncate the expansion as follows. (a) The Keplerian part is expanded up to the quadratic terms. The terms $h_{j_1,j_2}^{(\mathcal{T})}$ include: (b1) the linear terms in the actions $\underline{L}$, (b2) all terms up to degree 18 in the secular variables $(\underline{\xi},\underline{\eta})$, (b3) all terms up to the trigonometric degree 16 with respect to the angles $\underline{\lambda}$. Our choice of the limits will be motivated in the next section.

3 The secular model

We look now for a good description of the secular dynamics. A straightforward method would be to include in the unperturbed Hamiltonian also the average of the perturbation over the fast angles. However, it has been remarked by Robutel (see [45]) that the frequencies of the quasi–periodic flow given by this secular Hamiltonian (often called of order one in the masses) are quite different from the true ones. The reason lies in the effect of the mean motion quasi–resonance $5:2$. Therefore we look for an approximation of the secular Hamiltonian up to order two in the masses (see, e.g., [29], [31], [45], [35] and [34]). To this end we follow the approach in [37], carrying out two "Kolmogorov–like" normalization steps in order to eliminate the main perturbation terms depending on the fast angles $\underline{\lambda}$. We concentrate our attention on the resonant angles $2\lambda_1-5\lambda_2$, $\lambda_1-7\lambda_3$ and $3\lambda_1-5\lambda_2-7\lambda_3$, which are the most relevant ones for the dynamics. Our aim is to replace the orbit with zero eccentricity with a quasi periodic one that takes into account the effect of such resonances up to the second order in the masses. The procedure is a little cumbersome, and requires two main steps that we describe in the next two subsections.

3.1 Partial reduction of the perturbation

We emphasize that the Fourier expansion of the Hamiltonian (5) is generated just by terms due to two–body interactions, and so harmonics including more than two fast angles cannot appear. Thus, at first order in the masses only harmonics with the resonant angles $2\lambda_1-5\lambda_2$ and $\lambda_1-7\lambda_3$ do occur. Actually, harmonics with the resonant angle $3\lambda_1-5\lambda_2-7\lambda_3$ are generated by the first Kolmogorov–like transformation, but are of second order in the masses, and shall be removed by the second Kolmogorov–like transformation described in the next section.

Let us go into details. We denote by $\lceil f \rceil_{\underline{\lambda};K_F}$ the Fourier expansion of a function f truncated so as to include only its harmonics $\underline{k}\cdot\underline{\lambda}$ satisfying the restriction $0<|\underline{k}|\le K_F$. We also denote by $\langle\cdot\rangle_{\underline{\lambda}}$ the average with respect to the angles λ_1 , λ_2 , λ_3 . The canonical transformations are using the Lie series algorithm (see, e.g., [14]).

We set $K_F=8$ and transform the Hamiltonian (5) as $\hat{\mathcal{H}}^{(\mathcal{O}2)}=\exp\mathcal{L}_{\mu\chi_1^{(\mathcal{O}2)}}\,\mathcal{H}^{(\mathcal{T})}$ with

the generating function $\mu\,\chi_1^{(\mathcal{O}2)}(\underline{\lambda},\underline{\xi},\underline{\eta})$ determined by solving the equation

$$\sum_{j=1}^{3} n_j^* \frac{\partial\,\chi_1^{(\mathcal{O}2)}}{\partial\lambda_j} + \sum_{j_2=0}^{6} \left\lceil h_{0,j_2}^{(\mathcal{T})} \right\rceil_{\underline{\lambda};8} (\underline{\lambda},\underline{\xi},\underline{\eta}) = 0 \ . \tag{6}$$

Notice that, by definition, $\big\langle \lceil f \rceil_{\underline{\lambda};K_F} \big\rangle_{\underline{\lambda}} = 0$, which assures that equation (6) can be solved provided the frequencies $(n_1^*\,,\,n_2^*\,,\,n_3^*)$ are not resonant up to order 8, as it actually occurs in our planar model of the SJSU system.

The Hamiltonian $\hat{\mathcal{H}}^{(\mathcal{O}2)}$ has the same form of $\mathcal{H}^{(\mathcal{T})}$ in (5), with the functions $h_{j_1,j_2}^{(\mathcal{T})}$ replaced by new ones, that we denote by $\hat{h}_{j_1,j_2}^{(\mathcal{O}2)}$, generated by the expanding the Lie series $\exp\mathcal{L}_{\mu\chi_1^{(\mathcal{O}2)}}\mathcal{H}^{(\mathcal{T})}$ and by gathering all the terms having the same degree both in the fast actions and in the secular variables.

Now we perform a second canonical transformation $\mathcal{H}^{(\mathcal{O}2)} = \exp\mathcal{L}_{\mu\chi_2^{(\mathcal{O}2)}}\hat{\mathcal{H}}^{(\mathcal{O}2)}$, where the generating function $\mu\,\chi_2^{(\mathcal{O}2)}(\underline{L},\underline{\lambda},\underline{\xi},\underline{\eta})$ (which is linear with respect to $\underline{L}$) is determined by solving the equation

$$\sum_{j=1}^{3} n_j^* \frac{\partial\,\chi_2^{(\mathcal{O}2)}}{\partial\lambda_j} + \sum_{j_2=0}^{6} \left\lceil \hat{h}_{1,j_2}^{(\mathcal{O}2)} \right\rceil_{\underline{\lambda};8} (\underline{L},\underline{\lambda},\underline{\xi},\underline{\eta}) = 0 \ . \tag{7}$$

Again, the Hamiltonian $\mathcal{H}^{(\mathcal{O}2)}$ can be written in a form similar to (5), namely

$$\mathcal{H}^{(\mathcal{O}2)}(\underline{L},\underline{\lambda},\underline{\xi},\underline{\eta}) = \underline{n}^*\cdot\underline{L} + \sum_{j_1=2}^{\infty} h_{j_1,0}^{(\mathrm{Kep})}(\underline{L}) + \mu\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty} h_{j_1,j_2}^{(\mathcal{O}2)}(\underline{L},\underline{\lambda},\underline{\xi},\underline{\eta};\mu) \ . \tag{8}$$

where the new functions $h_{j_1,j_2}^{(\mathcal{O}2)}$ are calculated as previously explained for $\hat{h}_{j_1,j_2}^{(\mathcal{O}2)}$. Moreover, they still have the same dependence on their arguments as $h_{j_1,j_2}^{(\mathcal{T})}$ in (5).

If terms of second order in μ are neglected, then the Hamiltonian $\mathcal{H}^{(\mathcal{O}2)}$ possesses the secular 3–dimensional invariant torus $\underline{L}=\underline{0}$ and $\underline{\xi}=\underline{\eta}=\underline{0}$. Thus, in a small neighborhood of the origin of the fast actions and for small eccentricities the solutions of the system with Hamiltonian $\mathcal{H}^{(\mathcal{O}2)}$ differ from those of its average $\langle\mathcal{H}^{(\mathcal{O}2)}\rangle_{\underline{\lambda}}$ by a quantity $\mathcal{O}(\mu^2)$. In this sense the average of the Hamiltonian (8) approximates the real dynamics of the secular variables up to order two in the masses, and due to the choice $K_F=8$ takes into account the resonances $5:2$ between Jupiter and Saturn and $7:1$ between Jupiter and Uranus.

In this part of the calculation we produce a truncated series which is represented as a sum of monomials

$$c_{\underline{j},\underline{k},\underline{r},\underline{s}}\, L_1^{j_1} L_2^{j_2} L_3^{j_3}\, \xi_1^{r_1}\xi_2^{r_2}\xi_3^{r_3}\, \eta_1^{s_1}\eta_2^{s_2}\eta_3^{s_3}\, {\textstyle\frac{\sin}{\cos}}(k_1\lambda_1+k_2\lambda_2+k_3\lambda_3) \ .$$

The truncated expansion of $\mathcal{H}^{(\mathcal{O}2)}$ contains 94 109 751 such monomials. We truncate our expansion at degree 16 in the fast angles $\underline{\lambda}$ and at degree 18 in the slow variables $\underline{\xi}$, $\underline{\eta}$ (we shall justify this choice at the end of the next section).

3.2 Second approximation and reduction to the secular Hamiltonian

The huge number of coefficients determined till now does not allow us to continue by keeping all of them. Therefore, in view that we plan to consider the secular system, we perform a partial average by keeping only the main terms that contain the resonant angle $3\lambda_1 - 5\lambda_2 - 7\lambda_3$. More precisely, we first consider the reduced Hamiltonian

$$\left\langle \mathcal{H}^{(\mathcal{O}2)}\big|_{\underline{L}=\underline{0}} \right\rangle_{\underline{\lambda}} = \mu \sum_{j_2=0}^{\infty} \left\langle h_{0,j_2}^{(\mathcal{O}2)}(\underline{\xi}, \underline{\eta}; \mu) \right\rangle_{\underline{\lambda}} , \tag{9}$$

namely we set $\underline{L} = \underline{0}$, which results in replacing the orbit having zero eccentricity with a close invariant torus of the unperturbed Hamiltonian, and average $\mathcal{H}^{(\mathcal{O}2)}$ by removing all the Fourier harmonics depending on the angles. Next, we select in $\mathcal{H}^{(\mathcal{O}2)}$ the Fourier harmonics that contain the wanted resonant angle $3\lambda_1 - 5\lambda_2 - 7\lambda_3$ and add them to the Hamiltonian (9). Finally, we perform on the resulting Hamiltonian the second Kolmogorov–like step. With more detail, this is the procedure, which is an adaptation of a scheme already used in [35].

For $(j_1, j_2) \in \mathbb{N}^2$ we select the resonant terms

$$\begin{aligned} \mu^2 h_{j_1,j_2}^{(\mathrm{res})}(\underline{L}, \underline{\lambda}, \underline{\xi}, \underline{\eta}) = \mu \left\langle h_{j_1,j_2}^{(\mathcal{O}2)} \exp\big[-\mathrm{i}(3\lambda_1 - 5\lambda_2 - 7\lambda_3)\big] \right\rangle_{\underline{\lambda}} \exp\big[\mathrm{i}(3\lambda_1 - 5\lambda_2 - 7\lambda_3)\big] + \\ \mu \left\langle h_{j_1,j_2}^{(\mathcal{O}2)} \exp\big[\mathrm{i}(3\lambda_1 - 5\lambda_2 - 7\lambda_3)\big] \right\rangle_{\underline{\lambda}} \exp\big[-\mathrm{i}(3\lambda_1 - 5\lambda_2 - 7\lambda_3)\big] . \end{aligned} \tag{10}$$

Actually, this means that in our expression we just remove all monomials but the ones containing the wanted resonant angle. Using the selected terms we determine a generating function $\mu^2 \chi_1^{(\mathrm{res})}(\underline{\lambda}, \underline{\xi}, \underline{\eta})$ by solving the equation

$$\sum_{j=1}^{3} n_j^* \frac{\partial \chi_1^{(\mathrm{res})}}{\partial \lambda_j} + \sum_{j_2=0}^{9} h_{0,j_2}^{(\mathrm{res})}(\underline{\lambda}, \underline{\xi}, \underline{\eta}) = 0 . \tag{11}$$

Here we make the calculation faster by keeping only terms up to degree 9 in $(\underline{\xi}, \underline{\eta})$, this allows us to keep the more relevant resonant contributions. Then, still following the procedure outlined in [35], we calculate only the interesting part of the transformed Hamiltonian $\exp \mathcal{L}_{\mu^2 \chi_2^{(\mathrm{res})}} \exp \mathcal{L}_{\mu^2 \chi_1^{(\mathrm{res})}} \mathcal{H}^{(\mathcal{O}2)}$, namely we keep in the transformation only the part which is independent of all the fast variables $(\underline{L}, \underline{\lambda})$. This produces the secular Hamiltonian $\mathcal{H}^{(\mathrm{sec})}$, which satisfies the formal equation $\left\langle \exp \mathcal{L}_{\mu^2 \chi_2^{(\mathrm{res})}} \exp \mathcal{L}_{\mu^2 \chi_1^{(\mathrm{res})}} \mathcal{H}^{(\mathcal{O}2)} \right\rangle_{\underline{\lambda}} = \mathcal{H}^{(\mathrm{sec})} + \mathcal{O}(\|\underline{L}\|) + o(\mu^4)$, where

$$\begin{aligned} \mathcal{H}^{(\mathrm{sec})}(\underline{\xi}, \underline{\eta}) = \mu \sum_{j_2=0}^{\infty} \left\langle h_{0,j_2}^{(\mathcal{O}2)} \right\rangle_{\underline{\lambda}} + \mu^4 \Bigg\langle \frac{1}{2} \left\{ \chi_1^{(\mathrm{res})}, \mathcal{L}_{\mu^2 \chi_1^{(\mathrm{res})}} h_{2,0}^{(\mathrm{Kep})} \right\}_{\underline{L},\underline{\lambda}} + \\ \left\{ \chi_1^{(\mathrm{res})}, \sum_{j_2=0}^{\infty} h_{1,j_2}^{(\mathrm{res})} \right\}_{\underline{L},\underline{\lambda}} + \frac{1}{2} \left\{ \chi_1^{(\mathrm{res})}, \sum_{j_2=0}^{\infty} h_{0,j_2}^{(\mathrm{res})} \right\}_{\underline{\xi},\underline{\eta}} \Bigg\rangle_{\underline{\lambda}} . \end{aligned} \tag{12}$$

Here, we denoted by $\{\cdot,\cdot\}_{\underline{L},\underline{\lambda}}$ and $\{\cdot,\cdot\}_{\underline{\xi},\underline{\eta}}$ the terms of the Poisson bracket involving only the derivatives with respect the variables $(\underline{L},\underline{\lambda})$ and $(\underline{\xi},\underline{\eta})$, respectively.

The Hamiltonian so constructed is the secular one, describing the slow motion of eccentricities and perihelia. In view of D'Alembert rules (see, e.g., [44]), it contains only terms of even degree and so the lowest order significant term has degree 2. We have determined the power series expansion of the Hamiltonian up to degree 18 in the slow variables. In order to allow a comparison with other expansions, we reported our results up to degree 4 in $(\underline{\xi},\underline{\eta})$ in appendix A.

We close this section with a few remarks which justify our choice of the truncation orders. The limits on the expansions in the fast actions $\underline{L}$ have been illustrated at points (a) and (b1) at the end of section 2, and they are the smallest ones that are required in order to make the Kolmogorov–like normalization procedure significant. Since we want to keep the resonant angles $2\lambda_1-5\lambda_2$, $\lambda_1-7\lambda_3$ and $3\lambda_1-5\lambda_5-7\lambda_3$, we set the truncation order for Fourier series to 16, which is enough. The choice to truncate the expansion at degree 18 in the secular variables $(\underline{\xi},\underline{\eta})$ is somehow subtler. In view of D'Alembert rules the harmonics $2\lambda_1-5\lambda_2$ and $\lambda_1-7\lambda_3$ have coefficients of degree at least 3 and 6, respectively, in the secular variables. Furthermore, the resonant angle $3\lambda_1-5\lambda_5-7\lambda_3$ does not appear initially in the Hamiltonian, but is generated by Poisson bracket between the harmonics $2\lambda_1-5\lambda_2$ and $\lambda_1-7\lambda_3$, which produces monomials of degree 9 in $(\underline{\xi},\underline{\eta})$. Therefore, we decided to calculate the generating functions $\chi_1^{(\mathcal{O}2)}$ and $\chi_2^{(\mathcal{O}2)}$ up to degree 9 (recall equations (6) and (7)). Finally, in the second Kolmogorov–like step we want to keep the secular terms generated by the harmonic $3\lambda_1-5\lambda_5-7\lambda_3$, which are produced by Poisson bracket between monomials containing precisely this harmonic, and then the result has maximum degree 18 in $(\underline{\xi},\underline{\eta})$. This explains the final truncation order for the slow variables.

4 Stability of the secular Hamiltonian model

The lowest order approximation of the secular Hamiltonian $\mathcal{H}^{(\mathrm{sec})}$, namely its quadratic term, is essentially the one considered in the theory first developed by Lagrange (see [23]) and furtherly improved by Laplace (see [26], [27] and [28]) and by Lagrange himself (see [24], [25]). In modern language, we say that the origin of the reduced phase space (i.e., $(\underline{\xi},\underline{\eta})=(\underline{0},\underline{0})$) is an elliptic equilibrium point (for a review using a modern formalism, see sect. 3 of [4], where a planar model of our Solar System is considered).

It is well known that (under mild assumptions on the quadratic part of the Hamiltonian which are satisfied in our case) one can find a linear canonical transformation $(\underline{\xi},\underline{\eta})=\mathcal{D}(\underline{x},\underline{y})$ which diagonalizes the quadratic part of the Hamiltonian, so that we may write $\mathcal{H}^{(\mathrm{sec})}$ in the new coordinates as

$$H^{(0)}(\underline{x},\underline{y})=\sum_{j=0}^{3}\frac{\omega_j}{2}\left(x_j^2+y_j^2\right)+H_2^{(0)}(\underline{x},\underline{y})+H_4^{(0)}(\underline{x},\underline{y})+H_6^{(0)}(\underline{x},\underline{y})+\ldots\ , \tag{13}$$

where ω_j are the secular frequencies in the small oscillations limit and $H_{2s}^{(0)}$ is a homogeneous polynomial of degree $2s+2$ in $(\underline{x},\underline{y})$. The calculated values of $(\omega_1,\omega_2,\omega_3)$ in our

Table 2: Angular velocities $\underline{\omega}$ and initial conditions $(\underline{x}(0), \underline{y}(0))$ for our planar secular model about the motions of Jupiter, Saturn and Uranus. The frequency vector $\underline{\omega}$ refer to the harmonic oscillators approximation of the Hamiltonian $H^{(0)}$ (written in (13)) and its values are given in *rad/year* .

	$j=1$	$j=2$	$j=3$
ω_j	$-1.1212724892 \times 10^{-4}$	$-1.9688444678 \times 10^{-5}$	$-1.1134564418 \times 10^{-5}$
$x_j(0)$	$1.5407573458 \times 10^{-2}$	$-3.0574059274 \times 10^{-2}$	$1.1186486403 \times 10^{-2}$
$y_j(0)$	$-2.5320810665 \times 10^{-2}$	$-5.2728862107 \times 10^{-3}$	$6.0669645406 \times 10^{-3}$

case are reported in Table 2.

Thus, we are led to study the stability of the equilibrium for the Hamiltonian (13). As remarked in the introduction, perpetual stability in a neighbourhood of the equilibrium is assured in our case by Dirichlet's theorem because all frequencies have the same sign, that is negative in our case. Actually, a very rough evaluation of the size of the stability neighbourhood gives a value about 0.6 times the distance (from the origin) of the actual initial data of the planets. Such an estimate should certainly be improved by a more accurate calculation, i.e., by determining the stationary points of a function in 6 variables. However, we emphasize that our model is just a planar approximation of the true problem. If, for instance, one considers the spatial secular problem then the secular frequencies of the nodes have a positive sign, so that Dirichlet's theory does not apply any more. Thus, we think it is more interesting to investigate the stability of the equilibrium into the light of Nekhoroshev's theory.

4.1 Birkhoff's normal form

Following a quite standard procedure we proceed to construct the Birkhoff's normal form for the Hamiltonian (13) (see [5]; for an application of Nekhoroshev's theory see, e.g., [13]). This is a well known matter, thus we limit our exposition to a short sketch adapted to the present context.

The aim is to give the Hamiltonian the normal form at order r

$$H^{(r)}(\underline{x}, \underline{y}) = Z_0(\underline{\Phi}) + \ldots + Z_r(\underline{\Phi}) + \mathcal{F}^{(r)}_{r+1}(\underline{x}, \underline{y}) + \mathcal{F}^{(r)}_{r+2}(\underline{x}, \underline{y}) + \ldots \ , \tag{14}$$

where

$$\Phi_j = \frac{1}{2}\left(x_j^2 + y_j^2\right) \qquad \text{for } j = 1,\, 2,\, 3 \tag{15}$$

are the actions of the system, and Z_s for $s = 0, \ldots, r$ is a homogeneous polynomial of degree $s/2 + 1$ in $\underline{\Phi}$ and in particular it is zero for odd s. The un–normalized reminder terms $\mathcal{F}^{(r)}_s$, where $s > r$, are homogeneous polynomials of degree $s + 2$ in $(\underline{x}, \underline{y})$.

We proceed by induction. Assume that the Hamiltonian is in normal form up to a given order r, which is trivially true for $r = 0$, and determine a generating function $\chi^{(r+1)}$ and the normal form term Z_{r+1} , by solving the equation

$$\left\{\chi^{(r+1)}\, ,\, \underline{\omega} \cdot \underline{\Phi}\right\} + \mathcal{F}^{(r)}_{r+1}(\underline{x}, \underline{y}) = Z_{r+1}(\underline{\Phi}) \ . \tag{16}$$

Using the algorithm of Lie series transform, we can write the new Hamiltonian as $H^{(r+1)} = \exp \mathcal{L}_{\chi^{(r+1)}} H^{(r)}$. It is not difficult to show that $H^{(r+1)}$ has a form analogous to that written in (14) with new functions $\mathcal{F}_s^{(r+1)}$ of degree $s+2$ (where $s > r+1$) and the normal form part ending with Z_{r+1}, which is equal to zero if r is even (see, e.g., [16]). As usual when using the Lie series methods, we denote by $(\underline{x}, \underline{y})$ the new coordinates, so that the normal form $H^{(r)}$ possesses the approximate first integrals $\underline{\Phi}$ given by (15). By the way, the algorithm can be iterated up to the step r provided that the non-resonance condition

$$\underline{k} \cdot \underline{\omega} \neq 0 \qquad \forall\, \underline{k} \in \mathbb{Z}^3 \text{ such that } 0 < |\underline{k}| \leq r+2 \tag{17}$$

is fulfilled.

4.2 Study of the stability time

It is well known that the Birkhoff's normal form at any finite order r is convergent in some neighbourhood of the origin, but the analyticity radius shrinks to zero when $r \to \infty$. Thus, the best one can do is to look for stability for a finite but long time. We use the algorithm reported in [20], that we describe here.

Let us pick three positive numbers R_1, R_2, R_3 and consider a polydisk $\Delta_{\varrho\underline{R}}$ with center at the origin of $\mathbb{R}^6$ defined as

$$\Delta_{\varrho\underline{R}} = \left\{ (\underline{x}, \underline{y}) \in \mathbb{R}^6 : x_j^2 + y_j^2 \leq \varrho^2 R_j^2 \,,\ j = 1,\, 2,\, 3 \right\} \ ,$$

$\varrho > 0$ being a parameter. Let $\varrho_0 = \varrho/2$, and let $(\underline{x}_0, \underline{y}_0) \in \Delta_{\varrho_0\underline{R}}$ be the initial point of an orbit, so that one has $\Phi_j(0) = (x_j^2 + y_j^2)/2 \leq \varrho_0^2 R_j^2/2$. Therefore, there is $T(\varrho_0) > 0$ such that for $|t| \leq T(\varrho_0)$ we have $\underline{\Phi}(t) \leq \varrho^2 R_j^2/2$, and so also $(\underline{x}(t), \underline{y}(t)) \in \Delta_{\varrho\underline{R}}$. We call $T(\varrho_0)$ the *estimated stability time*, and our aim is to give a good estimate of it.

The key remark is that one has

$$\dot{\Phi}_j = \left\{ \Phi_j \,,\, H^{(r)} \right\} = \sum_{s=r+1}^{\infty} \left\{ \Phi_j \,,\, \mathcal{F}_s^{(r)} \right\} \simeq \left\{ \Phi_j \,,\, \mathcal{F}_{r+1}^{(r)} \right\} \qquad \text{for } j = 1,\, 2,\, 3 \ , \tag{18}$$

which holds true for an arbitrary normalization order r. This means that the time derivative of $\underline{\Phi}(t)$ is small, being $\mathcal{O}(\varrho^{r+3})$, so that the time $T(\varrho_0)$ may grow very large. The basis of Nekhoroshev's theory is that one can choose an optimal value of r as a function of ϱ_0 letting it to get larger and larger when $\varrho_0 \to 0$, so that $T(\varrho_0)$ grows faster than any power of $1/\varrho_0$. Here we give this argument an algorithmic form, thus producing an explicit estimate of $T(\varrho_0)$.

Let us write a homogeneous polynomial $f(\underline{x}, \underline{y})$ of degree s as

$$f(\underline{x}, \underline{y}) = \sum_{|\underline{j}|+|\underline{k}|=s} f_{\underline{j},\underline{k}} \underline{x}^{\underline{j}} \underline{y}^{\underline{k}} \ ,$$

where the multiindex notation $\underline{x}^{\underline{j}} \underline{y}^{\underline{k}} = x_1^{j_1} x_2^{j_2} x_3^{j_3} y_1^{k_1} y_2^{k_2} y_3^{k_3}$ has been used. We define the quantity $|f|_{\underline{R}}$ as

$$|f|_{\underline{R}} = \sum_{|\underline{j}|+|\underline{k}|=s} |f_{\underline{j},\underline{k}}| R_1^{j_1+k_1} R_2^{j_2+k_2} R_3^{j_3+k_3} \Theta_{j_1,k_1} \Theta_{j_2,k_2} \Theta_{j_3,k_3} \ , \quad \Theta_{j,k} = \sqrt{\frac{j^j k^k}{(j+k)^{j+k}}} \ . \tag{19}$$

We claim that for $\varrho \ge 0$ one has

$$\sup_{(\underline{x},\underline{y})\in\Delta_{\varrho\underline{R}}} \bigl|f(\underline{x},\underline{y})\bigr| \le \varrho^s |f|_{\underline{R}} \ . \tag{20}$$

The estimate is checked as follows. In the plane x_i, y_i consider a disk with radius R_i. Then inside the disk the inequality $|x_i^{j_i} y_i^{k_i}| \le R_i^{j_i+k_i}\Theta_{j_i,k_i}$ holds true. In fact, after having set $x_i = R_i\cos\theta$, $y_i = R_i\sin\theta$, one can easily check that $|\cos^{j_i}\theta\sin^{k_i}\theta| \le \Theta_{j_i,k_i}$. It is then straightforward to verify that for a monomial $\underline{x}^{\underline{j}}\underline{y}^{\underline{k}}$ of degree s one has

$$\sup_{(\underline{x},\underline{y})\in\Delta_{\varrho\underline{R}}} \bigl|\underline{x}^{\underline{j}}\underline{y}^{\underline{k}}\bigr| \le \varrho^s R_1^{j_1+k_1} R_2^{j_2+k_2} R_3^{j_3+k_3} \Theta_{j_1,k_1}\Theta_{j_2,k_2}\Theta_{j_3,k_3} \ .$$

The wanted inequality is just the sum of the contributions of all monomials.

Using (20) and (18) we can estimate

$$\sup_{(\underline{x},\underline{y})\in\Delta_{\varrho\underline{R}}} \bigl|\dot\Phi_j(x,y)\bigr| \le C\varrho^{r+3}\bigl|\{\Phi_j,\mathcal{F}_{r+1}^{(r)}\}\bigr|_{\underline{R}} \tag{21}$$

for $j = 1, 2, 3$ and with some $C \ge 1$. In fact, after having set ϱ smaller than the convergence radius of the remainder series $\mathcal{F}_s^{(r)}$ (where $s > r$), the above inequality is true for some C . In our calculation we set $C = 2$.

We come now to the calculation of the estimated stability time. Since $\Phi_j = \varrho^2 R_j^2/2$, we have $\dot\Phi_j = R_j^2\varrho\dot\varrho$ and, in view of inequality (21), also

$$\dot\varrho \le \frac{B_{r,j}}{R_j^2}\varrho^{r+2} \ , \quad B_{r,j} = C\bigl|\{\Phi_j,\mathcal{F}_{r+1}^{(r)}\}\bigr|_{\underline{R}} \ .$$

Thus a majorant of the function $\varrho(t)$ is given by the solution of the equation $\dot\varrho = B_{r,j}\varrho^{r+2}/R_j^2$. Setting ϱ_0 as the initial value we conclude that $\varrho(t) \le 2\varrho_0$ for all $|t| \le \tau(\varrho_0, r)$, where

$$\tau(\varrho_0,r) = \min_j \frac{R_j^2}{B_{r,j}}\int_{\varrho_0}^{2\varrho_0}\frac{d\sigma}{\sigma^{r+2}} = \min_j\left(1-\frac{1}{2^{r+1}}\right)\frac{R_j^2}{(r+1)B_{r,j}\,\varrho_0^{r+1}} \ . \tag{22}$$

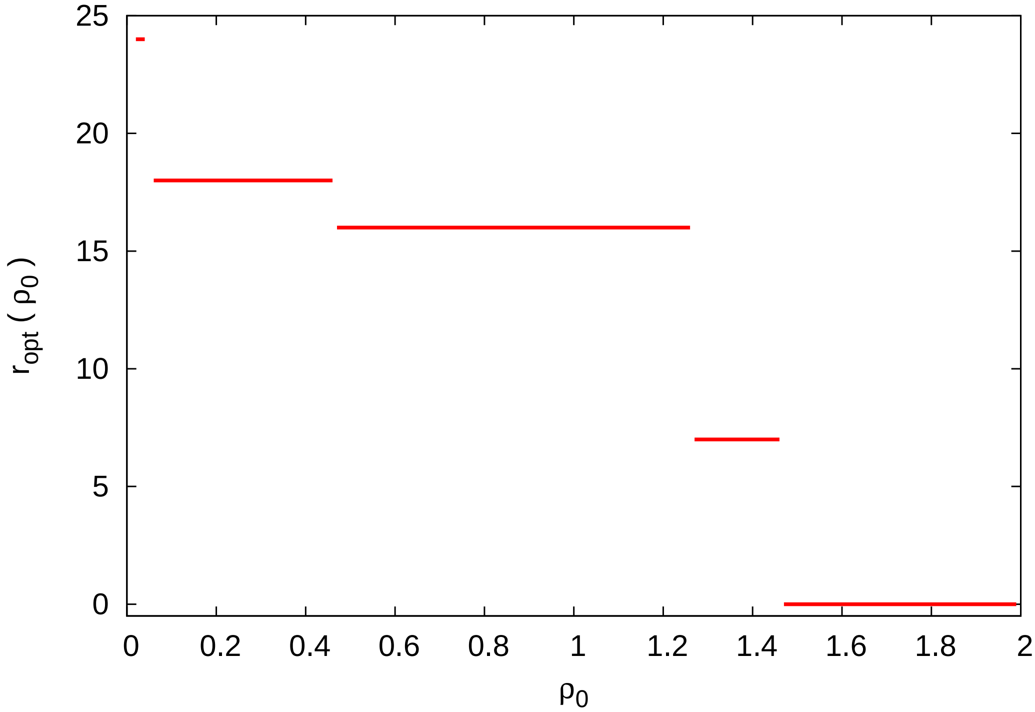
The latter estimate holds true for arbitrary normalization order r. Therefore we select an optimal order $r_{\rm opt}(\varrho_0)$ by looking for the maximum over r of $\tau(\varrho_0, r)$, thus getting

$$T(\varrho_0) = \max_r \tau(\varrho_0, 2\varrho_0, r) \ . \tag{23}$$

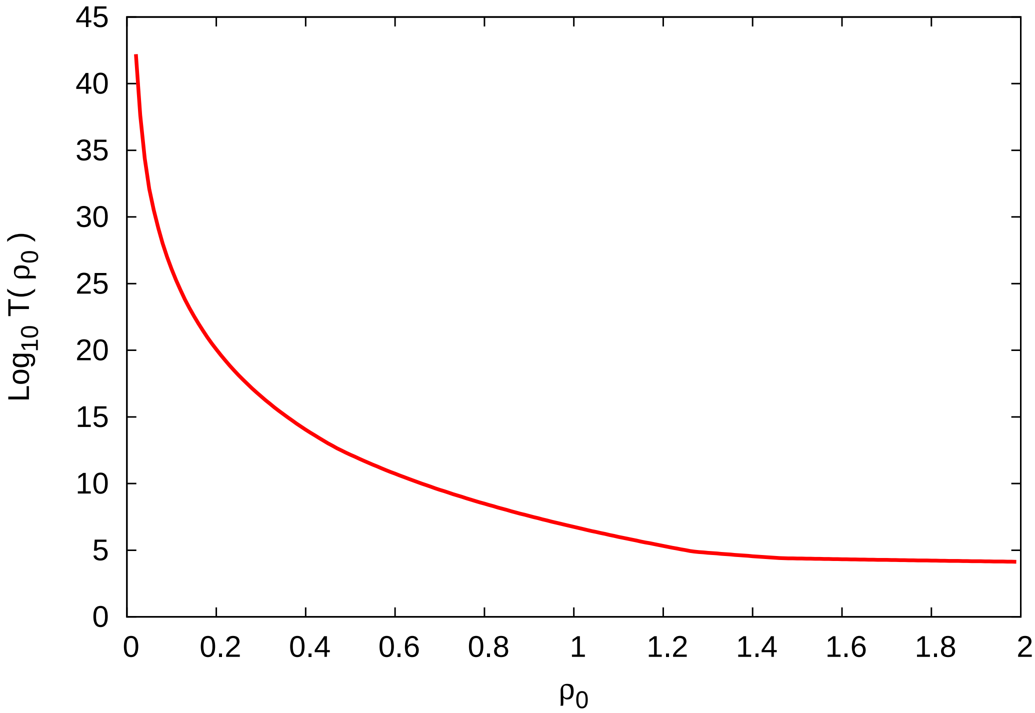
This is the best estimate of the stability time given by our algorithm.

4.3 Application to the SJSU system

We apply the algorithm of the previous section to the secular Hamiltonian $\mathcal{H}^{(\rm sec)}$ by explicitly performing the construction of Birkhoff's normal form up to order 30. Meanwhile also the first term of the remainder has been stored, so that the estimate for $\underline{\dot\Phi}$ is provided.



(a) Optimal normalization order



(b) Estimated stability time

Figure 1: Optimal normalization order $r_{\rm opt}$ and estimated stability time $T(\varrho_0)$ evaluated according to the algorithm of sect. 4.2. The time unit is the year. See text for more details.

The calculation of the estimated stability time is performed by setting

$$R_1 = 2.5558203988 \times 10^{-2}\ , \quad R_2 = 3.0601862602 \times 10^{-2}\ , \quad R_3 = 1.1223294461 \times 10^{-2}\ . \tag{24}$$

These values have been calculated as $R_j = \sqrt{x_j^2(0) + y_j^2(0)}$ where $x_j(0)$, $y_j(0)$ are the initial data reported in table 2, so that the initial point is on the border of the polydisk $\Delta_{\varrho \underline{R}}$ with $\varrho = 1$.

Finally we proceed to calculating the optimal normalization order $r_{\rm opt}(\varrho_0)$ and the estimated stability time $T(\varrho_0)$ as functions of ϱ_0 in an interval such that the optimal normalization order produced by our algorithm is less than 30. The results are reported in fig. 1. The fast increase of the time when ϱ decreases is evident from the graph. We also remark that for $\varrho_0 = 1$, which corresponds to the initial data for the planets, the normalization order is already $r_{\rm opt} = 16$. This shows that the mechanism of long time stability is already active. The estimated time with our algorithm is about 10^7 years for $\varrho_0 = 1$. This seems to be quite short both with respect to the age of the Solar System (which is estimated to be $\sim 5 \times 10^9$ years) and with respect to the numerical indications (10^{18} years). We shall comment on this point in the next section.

5 Conclusions and outlooks

In the framework provided by the Nekhoroshev's theorem, the present paper describes the first attempt to study the stability of a realistic model with more than two planets of our Solar System. As remarked at the end of the previous section we are not yet able to prove the stability for a time comparable to the age of our planetary system, even restricting ourselves to consider just the secular part of a planar approximation including the Sun, Jupiter, Saturn and Uranus. Nevertheless, we think that our results is meaningful in that it indicates that the phenomenon of exponential stability in Nekhoroshev sense may play an effective role for the Solar System, at least for the biggest planets. On the other hand, we stress that our result is not dramatically far from the goal of proving stability for the age of the Solar System: such a time is reached for $\varrho_0 \sim 0.7$. By the way, it may be worth to note that a similar result, with the same value of the radius, has been found in [20] where the spatial problem for the Sun–Jupiter–Saturn system is considered. Such a value of ϱ_0 appears to be not so small, especially if one recalls the rough estimates based on the first purely analytical proofs of the KAM theorem: in order to apply them to some model of our planetary system, the Jupiter mass should be smaller than that of a proton. Improvements are surely possible, and the relatively short history of the applications of the Nekhoroshev's type estimates to Celestial Mechanics has shown definitely more remarkable improvements than the one required here (e.g., compare [15] with [21]).

Some drawbacks are immediately evident. The most relevant one is that the estimate in (22) actually assumes that the perturbation constantly forces the worst possible evolution. This is clearly pessimistic, and justifies the striking difference with respect to the indication given by the numerical integrations. On the other hand, general perturbation method are essentially based on estimates that are often very crude. The explicit

calculation of normal forms and related quantities allows us to significantly improve our results, but the price is either a bigger and bigger computer power or more and more refined methods.

The natural question is whether there is a way to improve the present result. Our approach suggests that a better approximation of the true orbit could help a lot. This can be obtained, e.g., by first establishing the existence of a KAM torus close to the initial conditions of the planets, and then proving the stability in Nekhoroshev's sense in a neighbourhood of the torus that contains the initial data. Such an approach has been attempted in [19] for the Sun–Jupiter–Saturn case considering the full system, i.e., avoiding the approximation of the secular model. In that case the number of coefficients to be handled is so huge that the calculation can actually be performed only by introducing strong truncations on the expansions; this might artificially improve the results. Thus, some new idea is necessary, and this will be work for the future.

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A Expansion of the secular Hamiltonian of the planar SJSU system up to order 2 in the masses and 4 in eccentricities

Our secular model is represented by the Hamiltonian $\mathcal{H}^{(\mathrm{sec})}$, which is defined in (12). Here, we limit ourselves to report the expansion of $\mathcal{H}^{(\mathrm{sec})}$ up to degree 4 in $(\underline{\xi},\underline{\eta})$. Therefore, as a consequence of the D'Alembert rules, the terms related to the quasi–resonance $3\lambda_1 - 5\lambda_2 - 7\lambda_3$ do not give any contribution to the coefficients listed below. Thus, the following expansion of the rhs of (12) actually takes into account just $\mu\big\langle h_{0,2}^{(\mathcal{O}2)}\big\rangle_{\underline{\lambda}} + \mu\big\langle h_{0,4}^{(\mathcal{O}2)}\big\rangle_{\underline{\lambda}}$ (recall that $\mathcal{H}^{(\mathrm{sec})}$ contains just terms of even degree in its variables $(\underline{\xi},\underline{\eta})$). The calculation of the functions $h_{0,2}^{(\mathcal{O}2)}$ and $h_{0,4}^{(\mathcal{O}2)}$ is performed how it has been explained in subsect. 3.1.

$$
\begin{array}{ll}
\multicolumn{2}{l}{\mathcal{H}^{(\mathrm{sec})}(\underline{\xi},\underline{\eta}) =} \\
-\,2.0438249530856989\times 10^{-05}\,\xi_1^2 & +3.9042681895470743\times 10^{-05}\,\xi_1^1\,\xi_2^1 \\
+\,4.5005164146422330\times 10^{-07}\,\xi_1^1\,\xi_3^1 & -4.5352294644578622\times 10^{-05}\,\xi_2^2 \\
+\,1.9490388069796070\times 10^{-06}\,\xi_2^1\,\xi_3^1 & -5.6845848333331483\times 10^{-06}\,\xi_3^2 \\
-\,2.0438249530856989\times 10^{-05}\,\eta_1^2 & +3.9042681895470675\times 10^{-05}\,\eta_1^1\,\eta_2^1 \\
+\,4.5005164146422409\times 10^{-07}\,\eta_1^1\,\eta_3^1 & -4.5352294644578622\times 10^{-05}\,\eta_2^2 \\
+\,1.9490388069796070\times 10^{-06}\,\eta_2^1\,\eta_3^1 & -5.6845848333331441\times 10^{-06}\,\eta_3^2 \\
-\,1.0838003720922759\times 10^{-04}\,\xi_1^4 & +1.2014175808584642\times 10^{-03}\,\xi_1^3\,\xi_2^1 \\
+\,6.2045352476790196\times 10^{-07}\,\xi_1^3\,\xi_3^1 & -4.5563232782076350\times 10^{-03}\,\xi_1^2\,\xi_2^2
\end{array}
$$

$$
\begin{aligned}
&+ 8.8406443127175810 \times 10^{-07}\, \xi_1^2\, \xi_2^1\, \xi_3^1 && -9.7678628300067324 \times 10^{-06}\, \xi_1^2\, \xi_3^2 \\
&- 2.1676479523871672 \times 10^{-04}\, \xi_1^2\, \eta_1^2 && +1.2014125316196400 \times 10^{-03}\, \xi_1^2\, \eta_1^1\, \eta_2^1 \\
&+ 6.2157409102827665 \times 10^{-07}\, \xi_1^2\, \eta_1^1\, \eta_3^1 && -1.5832006427474584 \times 10^{-03}\, \xi_1^2\, \eta_2^2 \\
&+ 3.0033462029049336 \times 10^{-07}\, \xi_1^2\, \eta_2^1\, \eta_3^1 && -7.4173186653205456 \times 10^{-06}\, \xi_1^2\, \eta_3^2 \\
&+ 7.6046689202847869 \times 10^{-03}\, \xi_1^1\, \xi_2^3 && -2.4429460187142667 \times 10^{-06}\, \xi_1^1\, \xi_2^2\, \xi_3^1 \\
&+ 3.9912387029285291 \times 10^{-07}\, \xi_1^1\, \xi_2^1\, \xi_3^2 && +1.2014125316196422 \times 10^{-03}\, \xi_1^1\, \xi_2^1\, \eta_1^2 \\
&- 5.9464179266765730 \times 10^{-03}\, \xi_1^1\, \xi_2^1\, \eta_1^1\, \eta_2^1 && +5.8365071555190281 \times 10^{-07}\, \xi_1^1\, \xi_2^1\, \eta_1^1\, \eta_3^1 \\
&+ 7.6047082339419673 \times 10^{-03}\, \xi_1^1\, \xi_2^1\, \eta_2^2 && -1.6360484568891480 \times 10^{-06}\, \xi_1^1\, \xi_2^1\, \eta_2^1\, \eta_3^1 \\
&+ 2.2482538047243290 \times 10^{-07}\, \xi_1^1\, \xi_2^1\, \eta_3^2 && +2.6233130055605185 \times 10^{-05}\, \xi_1^1\, \xi_3^3 \\
&+ 6.2157409102827644 \times 10^{-07}\, \xi_1^1\, \xi_3^1\, \eta_1^2 && +5.8365071555190228 \times 10^{-07}\, \xi_1^1\, \xi_3^1\, \eta_1^1\, \eta_2^1 \\
&- 4.6671904570366227 \times 10^{-06}\, \xi_1^1\, \xi_3^1\, \eta_1^1\, \eta_3^1 && -8.0739065076924997 \times 10^{-07}\, \xi_1^1\, \xi_3^1\, \eta_2^2 \\
&+ 5.4429327654203341 \times 10^{-08}\, \xi_1^1\, \xi_3^1\, \eta_2^1\, \eta_3^1 && +2.6230652324380928 \times 10^{-05}\, \xi_1^1\, \xi_3^1\, \eta_3^2 \\
&- 4.8323841400859345 \times 10^{-03}\, \xi_2^4 && +2.9298658121783215 \times 10^{-05}\, \xi_2^3\, \xi_3^1 \\
&- 1.3020117317952433 \times 10^{-04}\, \xi_2^2\, \xi_3^2 && -1.5832006427474452 \times 10^{-03}\, \xi_2^2\, \eta_1^2 \\
&+ 7.6047082339419534 \times 10^{-03}\, \xi_2^2\, \eta_1^1\, \eta_2^1 && -8.0739065076924796 \times 10^{-07}\, \xi_2^2\, \eta_1^1\, \eta_3^1 \\
&- 9.6647220999081795 \times 10^{-03}\, \xi_2^2\, \eta_2^2 && +2.9299286278904711 \times 10^{-05}\, \xi_2^2\, \eta_2^1\, \eta_3^1 \\
&- 7.7905487286026464 \times 10^{-05}\, \xi_2^2\, \eta_3^2 && +1.9476359726545943 \times 10^{-04}\, \xi_2^1\, \xi_3^3 \\
&+ 3.0033462029049320 \times 10^{-07}\, \xi_2^1\, \xi_3^1\, \eta_1^2 && -1.6360484568891460 \times 10^{-06}\, \xi_2^1\, \xi_3^1\, \eta_1^1\, \eta_2^1 \\
&+ 5.4429327654202779 \times 10^{-08}\, \xi_2^1\, \xi_3^1\, \eta_1^1\, \eta_3^1 && +2.9299286278906453 \times 10^{-05}\, \xi_2^1\, \xi_3^1\, \eta_2^2 \\
&- 1.0427780546265602 \times 10^{-04}\, \xi_2^1\, \xi_3^1\, \eta_2^1\, \eta_3^1 && +1.9476665827159131 \times 10^{-04}\, \xi_2^1\, \xi_3^1\, \eta_3^2 \\
&- 2.0277494194124600 \times 10^{-04}\, \xi_3^4 && -7.4173186653203601 \times 10^{-06}\, \xi_3^2\, \eta_1^2 \\
&+ 2.2482538047243565 \times 10^{-07}\, \xi_3^2\, \eta_1^1\, \eta_2^1 && +2.6230652324380681 \times 10^{-05}\, \xi_3^2\, \eta_1^1\, \eta_3^1 \\
&- 7.7905487286035477 \times 10^{-05}\, \xi_3^2\, \eta_2^2 && +1.9476665827159768 \times 10^{-04}\, \xi_3^2\, \eta_2^1\, \eta_3^1 \\
&- 4.0555535091919988 \times 10^{-04}\, \xi_3^2\, \eta_3^2 && -1.0838003720922736 \times 10^{-04}\, \eta_1^4 \\
&+ 1.2014175808584629 \times 10^{-03}\, \eta_1^3\, \eta_2^1 && +6.2045352476790196 \times 10^{-07}\, \eta_1^3\, \eta_3^1 \\
&- 4.5563232782075760 \times 10^{-03}\, \eta_1^2\, \eta_2^2 && +8.8406443127175704 \times 10^{-07}\, \eta_1^2\, \eta_2^1\, \eta_3^1 \\
&- 9.7678628300066206 \times 10^{-06}\, \eta_1^2\, \eta_3^2 && +7.6046689202847939 \times 10^{-03}\, \eta_1^1\, \eta_2^3 \\
&- 2.4429460187142612 \times 10^{-06}\, \eta_1^1\, \eta_2^2\, \eta_3^1 && +3.9912387029285359 \times 10^{-07}\, \eta_1^1\, \eta_2^1\, \eta_3^2 \\
&+ 2.6233130055604931 \times 10^{-05}\, \eta_1^1\, \eta_3^3 && -4.8323841400860802 \times 10^{-03}\, \eta_2^4 \\
&+ 2.9298658121781443 \times 10^{-05}\, \eta_2^3\, \eta_3^1 && -1.3020117317952618 \times 10^{-04}\, \eta_2^2\, \eta_3^2 \\
&+ 1.9476359726546422 \times 10^{-04}\, \eta_2^1\, \eta_3^3 && -2.0277494194122486 \times 10^{-04}\, \eta_3^4 \\
&+ o\big(\|(\underline{\xi}, \underline{\eta})\|^4\big)\ .
\end{aligned}
$$

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